

Hyperplane Neural Codes and the Polar Complex



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Abstract Hyperplane codes are a class of convex codes that arise as the output of a one layer feed-forward neural network. Here we establish several natural properties of stable hyperplane codes in terms of the *polar complex* of the code, a simplicial complex associated to any combinatorial code. We prove that the polar complex of a stable hyperplane code is shellable and show that most currently known properties of hyperplane codes follow from the shellability of the appropriate polar complex.

1 Introduction

Combinatorial codes, i.e. subsets of the Boolean lattice, naturally arise as outputs of neural networks. A *codeword* $\sigma \subseteq [n] \stackrel{\text{def}}{=} \{1, \dots, n\}$ represents an allowed subset of co-active neurons, while a *code* is a collection $C \subseteq 2^{[n]}$ of codewords. Combinatorial codes in a number of areas of the brain are often *convex*, i.e. they arise as an intersection pattern of convex sets in a Euclidean space [17, 20, 24]. The combinatorial code of a one-layer feedforward neural network is also convex, as it arises as the intersection patterns of half-spaces [13, 25]. It is well-known that a two-layer feedforward network can approximate any measurable function [11, 19], and thus may produce any combinatorial code. In contrast, the codes of one-layer feedforward networks are not well-understood. The intersection lattices of affine hyperplane arrangements have been studied in the oriented matroid literature [1, 2, 4]. However, combinatorial codes contain less detailed information than

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oriented matroids, and the precise relationship is not clear. We are motivated by the following question: How can one determine if a given combinatorial code is realizable as the output of a one-layer feedforward neural network?

We study stable hyperplane codes, codes that arise from the intersection patterns of half-spaces that are stable under certain small perturbations. The paper is organized as follows. Relevant background and definitions are provided in Sect. 2. In Sect. 3, we establish a number of obstructions that prevent a combinatorial code from being a stable hyperplane code. In Sect. 4, we show that all but one of the currently known obstructions to being a stable hyperplane code are subsumed by the condition that the *polar complex* of the code, defined in Sect. 2.3, is shellable. Lastly, in Sect. 6 we show how techniques from commutative algebra can be used to computationally detect the presence of these obstructions.

2 Background

2.1 Stable Hyperplane Codes

We call a collection $\mathcal{U} = \{U_i\}$ of n subsets $U_i \subseteq X$ of a set X an *arrangement* (\mathcal{U}, X) . Note that we do *not* require that $\bigcup_{i \in [n]} U_i = X$.

Definition 1 For $\sigma \subseteq [n]$, let $A_\sigma^\mathcal{U}$ denote the *atom* of (\mathcal{U}, X)

$$A_\sigma^\mathcal{U} \stackrel{\text{def}}{=} \left(\bigcap_{i \in \sigma} U_i \right) \setminus \bigcup_{j \notin \sigma} U_j \subseteq X, \quad \text{where } A_\emptyset^\mathcal{U} \stackrel{\text{def}}{=} X \setminus \bigcup_{i \in [n]} U_i.$$

The *code* of the arrangement (\mathcal{U}, X) is defined as

$$\text{code}(\mathcal{U}, X) \stackrel{\text{def}}{=} \{\sigma \subseteq [n] \text{ such that } A_\sigma^\mathcal{U} \neq \emptyset\} \subseteq 2^{[n]}.$$

A *realization* of a code C is an arrangement (\mathcal{U}, X) such that $C = \text{code}(\mathcal{U}, X)$. The *simplicial complex of the code*, denoted $\Delta(C)$, is the closure of C under inclusion:

$$\Delta(C) \stackrel{\text{def}}{=} \{\tau \mid \tau \subseteq \sigma \text{ for some } \sigma \in C\}.$$

Note that for $C = \text{code}(\mathcal{U}, X)$, the simplicial complex of the code is equal to the *nerve* of the corresponding cover:

$$\Delta(\text{code}(\mathcal{U}, X)) = \text{nerve}(\mathcal{U}) \stackrel{\text{def}}{=} \left\{ \sigma \subseteq [n] \mid \bigcap_{i \in \sigma} U_i \neq \emptyset \right\}.$$

A natural class of codes that arises in the context of neural networks is the class of hyperplane codes [13]. A hyperplane is a level set $H = \{x \in \mathbb{R}^d \mid w \cdot x - h = 0\}$

of a non-constant affine function. An *oriented* hyperplane partitions \mathbb{R}^d into three pieces: $\mathbb{R}^d = H^+ \sqcup H \sqcup H^-$, where H^\pm are the open half-spaces, e.g. $H^+ \stackrel{\text{def}}{=} \{x \in \mathbb{R}^d \mid w \cdot x - h > 0\}$.

Definition 2 A code $C \subseteq 2^{[n]}$ is a *hyperplane code*, if there exists an open convex subset $X \subseteq \mathbb{R}^d$ and a collection $\mathcal{H} = \{H_1^+, \dots, H_n^+\}$ of open half-spaces such that $C = \text{code}(\{H_i^+ \cap X\}, X)$. With a slight abuse of notation, we denote this arrangement of subsets of X by (\mathcal{H}, X) , thus $\text{code}(\mathcal{H}, X) = \text{code}(\{H_i^+ \cap X\}, X)$.

Hyperplane codes are produced by one-layer feedforward neural networks [13], where the convex set X is often the positive orthant $\mathbb{R}_{\geq 0}^d$. A well-behaved subset of hyperplane codes are the *stable* hyperplane codes. Informally, these are codes that are preserved under small perturbations of the hyperplanes and the convex set X . These perturbations correspond to perturbations of the parameters of the neural network [25], i.e. the vectors $(w_i, h_i) \in \mathbb{R}^d \times \mathbb{R}$ in our context. Thus, we restrict our attention to the class of stable hyperplane codes.

Definition 3 An arrangement (\mathcal{H}, X) is *stable* if X is open and convex, and the hyperplanes have *generic intersections* in X , that is, if $X \cap H_\sigma \stackrel{\text{def}}{=} X \cap \bigcap_{i \in \sigma} H_i \neq \emptyset$, then $\dim H_\sigma = d - |\sigma|$.

We call a code C a *stable hyperplane code* if there exists a stable arrangement (\mathcal{H}, X) such that $C = \text{code}(\mathcal{H}, X)$.

Stable arrangements are robust to noise in the sense that all atoms have nonzero measure.

Lemma 1 *If (\mathcal{H}, X) is a stable arrangement, then every nonempty atom $A_\sigma^{\mathcal{U}}$ of the cover $\mathcal{U} = \{H_i^+ \cap X\}$ has a nonempty interior.*

Proof Let A_σ be a nonempty atom of the stable arrangement (\mathcal{H}, X) and consider a point $x \in A_\sigma$. Let $\tau = \{j \mid x \in H_j\}$ index the set of hyperplanes on which x lies. Then x has an open neighborhood V inside $X \cap (\bigcap_{i \in \sigma} H_i^+) \cap (\bigcap_{j \notin \sigma \cup \tau} H_j^-)$. By genericity, the set $\{w_i \mid i \in \tau\}$ is linearly independent. Therefore, there exists some $v \in \mathbb{R}^d$ such that $w_i \cdot v < 0$ for all $i \in \tau$. For sufficiently small $\varepsilon > 0$, $y = x + \varepsilon v \in V$; therefore for any $i \in \tau$,

$$w_i \cdot y - h_i = w_i \cdot (x + \varepsilon v) - h_i = w_i \cdot \varepsilon v < 0,$$

and thus $y \in X \cap (\bigcap_{i \in \sigma} H_i^+) \cap (\bigcap_{j \notin \sigma} H_j^-)$, which is the interior of A_σ . □

Example 1 The code $C_1 = \{1, 12, 123, 2, 23\}$ is a stable hyperplane code; a realization is illustrated in Fig. 1a. To avoid notational clutter, we adopt the convention of writing sets without brackets or commas, so the set $\{1, 2\}$ is written 12.

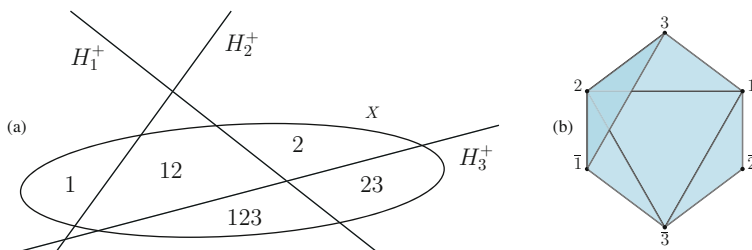


Fig. 1 (a) Stable arrangement (\mathcal{H}, X) with atoms labeled by their corresponding codewords. (b) The polar complex $\Gamma(\text{code}(\mathcal{H}, X))$, defined in Sect. 2.3

2.2 Bitflips and Stable Hyperplane Codes

The abelian group $(\mathbb{Z}_2)^n$ acts on $2^{[n]}$ by “flipping bits” of codewords. Each generator $\mathbf{e}_i \in (\mathbb{Z}_2)^n$ acts by flipping the i -th bit, i.e.

$$\mathbf{e}_i \cdot \sigma \stackrel{\text{def}}{=} \begin{cases} \sigma \cup i & \text{if } i \notin \sigma \\ \sigma \setminus i & \text{if } i \in \sigma. \end{cases}$$

This action extends to the action of $(\mathbb{Z}_2)^n$ on codes, with $g \cdot C = \{g \cdot \sigma \mid \sigma \in C\}$. The group $(\mathbb{Z}_2)^n$ also acts on oriented hyperplane arrangements. Here each generator \mathbf{e}_i acts by reversing the orientation of the i -th hyperplane:

$$\mathbf{e}_i \cdot H_j^+ \stackrel{\text{def}}{=} \begin{cases} H_j^+ & \text{if } i \neq j \\ H_j^- & \text{if } i = j. \end{cases}$$

One might hope that applying bitflips commutes with taking the code of a hyperplane arrangement, but this is not true for arbitrary hyperplane codes.

Example 2 Consider $H_1^+, H_2^+, H_3^+ \subseteq \mathbb{R}^2$, with $H_1^+ = \{x + y > 0\}$, $H_2^+ = \{x - y > 0\}$, and $H_3^+ = \{x > 0\}$, illustrated in Fig. 2a. By inspection, $C_2 = \text{code}(\mathcal{H}, \mathbb{R}^2)$ has codewords $\{\emptyset, 1, 13, 123, 23, 2\}$. Meanwhile,

$$\text{code}(\mathbf{e}_3 \cdot \mathcal{H}, \mathbb{R}^2) = \{3, 13, 1, 12, 2, 23, \emptyset\} = \mathbf{e}_3 \cdot \text{code}(\mathcal{H}, \mathbb{R}^2) \cup \{\emptyset\}.$$

The extra codeword appears because after flipping hyperplane H_3 , the origin no longer belongs to the same atom as the points to its left, and thus produces a new codeword, see Fig. 2b.

Nevertheless, the group action *does* commute with taking the code of a stable hyperplane arrangement.

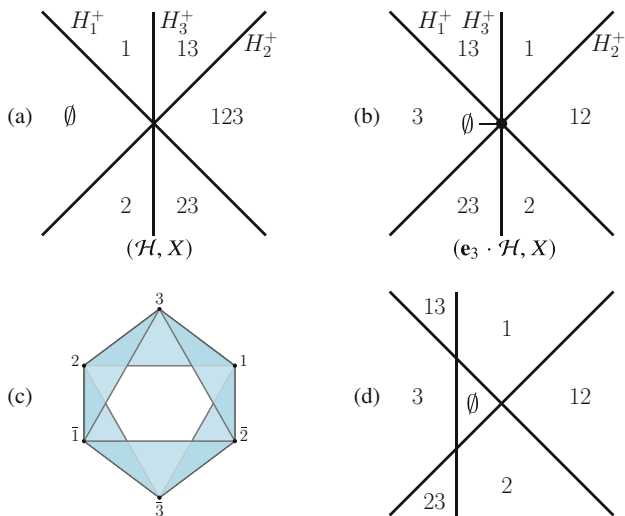


Fig. 2 (a, b) The action of $(\mathbb{Z}_2)^n$ does not necessarily commute with taking the code of a non-stable hyperplane arrangement. (c) The polar complex $\Gamma(\text{code}(\mathcal{H}, X))$ for the arrangement in panel (a) is an octahedron missing two opposite faces. (d) A stable realization of $\text{code}(e_3 \cdot \mathcal{H}, X)$, obtained from panel (b) by translating H_3 to the left

Proposition 1 *If (\mathcal{H}, X) is a stable arrangement, then for every $g \in (\mathbb{Z}_2)^n$, $(g \cdot \mathcal{H}, X)$ is also a stable arrangement and*

$$\text{code}(g \cdot \mathcal{H}, X) = g \cdot \text{code}(\mathcal{H}, X). \tag{1}$$

Proof Since the action of $(\mathbb{Z}_2)^n$ does not change the hyperplanes H_i (only their orientation) nor the set X , the stability is preserved. By Lemma 1, each atom of (\mathcal{H}, X) has a nonempty interior; this interior is not changed by reorientation of the hyperplanes. Thus, atoms are neither created nor destroyed by reorienting hyperplanes in a stable arrangement; only their labels change, and $\text{code}(g \cdot \mathcal{H}, X) = g \cdot \text{code}(\mathcal{H}, X)$. \square

2.3 The Polar Complex

The invariance (1) of the class of stable hyperplane codes under the $(\mathbb{Z}_2)^n$ action makes it natural to consider a simplicial complex whose structure is preserved by bitflips. The simplicial complex of the code is insufficient for this purpose: for any nontrivial code $C \subseteq 2^{[n]}$ with a nonempty codeword, the simplicial complexes of the codes in the $(\mathbb{Z}_2)^n$ -orbit of C will include the full simplex on n vertices, regardless of the structure of $\Delta(C)$.

We denote by $[n] \stackrel{\text{def}}{=} \{1, \dots, n\}$ and $\overline{[n]} \stackrel{\text{def}}{=} \{\overline{1}, \dots, \overline{n}\}$ two separate copies of the vertex set. Given a code $C \subseteq 2^{[n]}$, define the *polar complex*, $\Gamma(C)$, as a pure $(n - 1)$ -dimensional simplicial complex on vertex set $[n] \sqcup \overline{[n]}$ with facets in bijection with the codewords of C .

Definition 4 Let $C \subseteq 2^{[n]}$ be a combinatorial code. For every codeword $\sigma \in C$ denote

$$\Sigma(\sigma) \stackrel{\text{def}}{=} \{i \mid i \in \sigma\} \sqcup \{\overline{i} \mid i \notin \sigma\} = \sigma \sqcup \overline{[n] \setminus \sigma}$$

and define the *polar complex of C* as

$$\Gamma(C) \stackrel{\text{def}}{=} \Delta(\{\Sigma(\sigma) \mid \sigma \in C\}).$$

Continuing Example 1, the polar complex of $C_1 = \{1, 12, 123, 2, 23\}$ is given by $\Gamma(C_1) = \Delta(\{\overline{123}, 123, 123, \overline{123}, \overline{123}\})$. It is depicted in Fig. 1b as a subcomplex of the octahedron. The polar complex $\Gamma(2^{[3]})$ consists of the eight boundary faces of the octahedron; generally, the polar complex of the code consisting of all 2^n codewords on n vertices is the boundary of the n -dimensional cross-polytope.

The polar complex of code C_2 in Example 2 is depicted in Fig. 2c. Note that it follows from Theorem 4 that C_2 is *not* a stable hyperplane code, due to the structure of $\Gamma(C_2)$. In contrast, while Fig. 2b depicts a non-stable arrangement, the code of that arrangement has a stable realization depicted in Fig. 2d.

The action of the bitflips $(\mathbb{Z}_2)^n$ on the boolean lattice induces an action on the facets of the polar complex, so that $g \cdot \Sigma(\sigma) = \Sigma(g \cdot \sigma)$. In particular, $\Gamma(g \cdot C) = g \cdot \Gamma(C)$, and the complex $\Gamma(g \cdot C)$ is isomorphic to $\Gamma(C)$. The Stanley-Reisner ideal of $\Gamma(C)$ is closely related to the neural ideal, defined in [9]; this will be elaborated in Sect. 6. Moreover, in the case of stable hyperplane codes, $\Gamma(C)$ has a simple description as the nerve of a cover:

Lemma 2 *If $C = \text{code}(\mathcal{H}, X)$ is the code of a stable hyperplane arrangement, then*

$$\Gamma(C) = \text{nerve}(\{H_i^+ \cap X, H_i^- \cap X\}_{i \in [n]}) \tag{2}$$

Proof Consider a maximal face $\Sigma(\sigma) \in \Gamma(C)$. By Lemma 1, A_σ has nonempty interior given by $X \cap \bigcap_{i \in \sigma} H_i^+ \cap \bigcap_{j \notin \sigma} H_j^-$, hence $\Sigma(\sigma) \in \text{nerve}(\{H_i^+ \cap X, H_i^- \cap X\}_{i \in [n]})$. Likewise, if F is maximal in the complex $\text{nerve}(\{H_i^+ \cap X, H_i^- \cap X\}_{i \in [n]})$, the subset consisting of unbarred vertices in F is a codeword as the corresponding atom is nonempty. □

3 Obstructions for Hyperplane Codes

Here we describe several major *hyperplane obstructions*, the properties of a combinatorial code that are necessary for it to be realized by a stable hyperplane arrangement.

3.1 Local Obstructions and Bitflips

A larger class of codes that arises in the neuroscience context are the open convex codes [7–9, 13]. A code $C \subseteq 2^{[n]}$ is called *open convex* if there exists a collection \mathcal{U} of n open and convex sets $U_i \subseteq X \subseteq \mathbb{R}^d$, such that $C = \text{code}(\mathcal{U}, X)$. Not every combinatorial code is convex. One obstruction to being an open convex code stems from an analogue of the nerve lemma [3], recently proved in [6]; see also [21].

Recall the *link* of a face σ in a simplicial complex Δ is the subcomplex defined by

$$\text{link}_\sigma \Delta \stackrel{\text{def}}{=} \{\nu \in \Delta \mid \sigma \cap \nu = \emptyset, \sigma \cup \nu \in \Delta\}.$$

When $\sigma \notin \text{code}(\mathcal{U}, X)$, yet $\sigma \in \text{nerve}(\mathcal{U})$, the subset $U_\sigma \stackrel{\text{def}}{=} \bigcap_{i \in \sigma} U_i$ is covered by the collection of sets $\{U_j \cap U_\sigma\}_{j \notin \sigma}$. It is easy to see that in this situation,

$$\text{link}_\sigma \text{nerve}(\mathcal{U}) = \text{nerve}(\{U_j \cap U_\sigma\}_{j \notin \sigma}),$$

see e.g. [7, 8, 13].

Definition 5 A pair of faces (σ, τ) of a simplicial complex Δ is a *free pair* if τ is a facet of Δ , $\sigma \subsetneq \tau$, and $\sigma \not\subseteq \tau'$ for any other facet $\tau' \neq \tau$. The simplicial complex

$$\text{del}_\sigma \Delta \stackrel{\text{def}}{=} \{\nu \in \Delta \mid \nu \not\supseteq \sigma\}$$

is called the *collapse of Δ along σ* , and is denoted as $\Delta \searrow_\sigma \text{del}_\sigma \Delta$. If a finite sequence of collapses of Δ results in a new complex Δ' , we write $\Delta \searrow \Delta'$. If $\Delta \searrow \{\}$, we say Δ is *collapsible*.

Note that the *irrelevant simplicial complex* $\{\emptyset\}$, consisting of a single empty face, is *not* collapsible, as there is no other face properly contained in \emptyset . However, the *void complex* $\{\}$ with no faces is collapsible.

Lemma 3 ([6, Lemma 5.9], [21]) *For any collection $\mathcal{U} = \{U_1, \dots, U_n\}$ of open convex sets $U_i \subset \mathbb{R}^d$ whose union $\bigcup_{i \in [n]} U_i$ is also convex, its nerve, $\text{nerve}(\mathcal{U})$, is collapsible.*

Corollary 1 ([6, Theorem 5.10]) *Let $C = \text{code}(\mathcal{U}, X)$ with each $U_i \subseteq X \subseteq \mathbb{R}^d$ open and convex. Then $\text{link}_\sigma \Delta(C)$ is collapsible for every nonempty $\sigma \in \Delta(C) \setminus C$.*

The last observation provides a “local obstruction” for a code C being an open convex code: if a non-empty $\sigma \in \Delta(C) \setminus C$ has a non-collapsible link, then C is nonconvex. It had been previously known (see, for example, [13, Theorem 3]) that $\text{link}_\sigma \Delta(C)$ is contractible under the hypotheses of Corollary 1. Since collapsibility implies contractibility but not vice versa, we refer to a face $\sigma \in \Delta(C) \setminus C$ with non-collapsible link as a *strong* local obstruction; if $\text{link}_\sigma \Delta(C)$ is non-contractible, we refer to σ as a *weak* local obstruction.

Half-spaces are convex, thus local obstructions to being a convex code are also obstructions to being a hyperplane code. Therefore Proposition 1 implies a much stronger statement. Not only are local obstructions in C forbidden, we must also exclude local obstructions in $g \cdot C$ for all bitflips $g \in (\mathbb{Z}_2)^n$, since $g \cdot C$ is also a stable hyperplane code. We make this precise below.

Definition 6 Let $g \in (\mathbb{Z}_2)^n$ and $\tau \subseteq [n]$ be a pair such that $\text{link}_\tau \Delta(g \cdot C)$ is not collapsible (respectively, contractible) and $\tau \notin g \cdot C$. Then (g, τ) is called a *strong* (resp. *weak*) *bitflip local obstruction*.

Theorem 1 (Bitflip Local Property) *Suppose C is a stable hyperplane code. Then C has no strong bitflip local obstructions.*

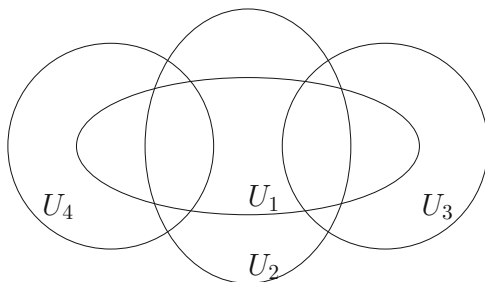
Proof Halfspaces are convex, thus C has no strong local obstructions. By Proposition 1, $g \cdot C$ is a stable hyperplane code for all $g \in (\mathbb{Z}_2)^n$. Hence, $g \cdot C$ has no strong local obstructions. \square

The nomenclature of “weak” and “strong” local obstructions signifies that a code with no strong local obstructions has no weak local obstructions, but generally not vice-versa. In particular, a stable hyperplane code also has no weak bitflip local obstructions.

Example 3 The code $C_3 = \{\emptyset, 2, 3, 4, 12, 13, 14, 23, 24, 123, 124\}$ is realizable by open convex sets in \mathbb{R}^2 (see Fig. 3), and thus it cannot have local obstructions to convexity. Flipping bit 2 yields

$$e_2 \cdot C_3 = \{2, \emptyset, 23, 24, 1, 123, 124, 3, 4, 13, 14\}.$$

Fig. 3 An open convex realization of C_3 with $X = \mathbb{R}^2$



The new simplicial complex $\Delta(\mathbf{e}_2 \cdot C_3)$ has facets 123 and 124. The edge 12 is *not* in the code and $\text{link}_{12}\Delta(\mathbf{e}_2 \cdot C_3)$ is two vertices; therefore, $(\mathbf{e}_2, 12)$ is a bitflip local obstruction and C_3 is not a stable hyperplane code.

It is worth highlighting an essential feature of the polar complex that makes it a natural tool for studying hyperplane codes, in light of the bitflip local property. For every $g \in (\mathbb{Z}_2)^n$, the simplicial complex $\Delta(g \cdot C)$ is isomorphic to an induced subcomplex of $\Gamma(C)$: Let σ denote the support of g and define

$$\Gamma(C)|_{([n] \setminus \sigma) \sqcup \bar{\sigma}} \stackrel{\text{def}}{=} \{F \in \Gamma(C) \mid F \subseteq ([n] \setminus \sigma) \sqcup \bar{\sigma}\}.$$

Then $\Gamma(C)|_{([n] \setminus \sigma) \sqcup \bar{\sigma}} \cong \Delta(g \cdot C)$, with the isomorphism given by “ignoring the bars,” i.e. $i \mapsto i$ for $i \in [n] \setminus \sigma$ and $\bar{j} \mapsto j$ for $j \in \sigma$. Thus we can find bitflip local obstructions directly in the polar complex as follows.

Proposition 2 *Let $C \subseteq 2^{[n]}$ be a code, $g \in (\mathbb{Z}_2)^n$ with σ its support, and let $\tau \subseteq [n]$. Then (g, τ) is a bitflip local obstruction for C if and only if*

$$g \cdot \tau \sqcup \overline{[n] \setminus g \cdot \tau} \notin \Gamma(C) \quad \text{and} \quad \text{link}_{g \cdot \tau} \Gamma(C)|_{([n] \setminus \sigma) \sqcup \bar{\sigma}} \text{ is not collapsible.}$$

Proof Note that $g \cdot \tau \sqcup \overline{[n] \setminus g \cdot \tau} \notin \Gamma(C)$ if and only if $\tau \notin g \cdot C$. The complex $\Gamma(C)|_{([n] \setminus \sigma) \sqcup \bar{\sigma}}$ is isomorphic to $\Delta(g \cdot C)$, and

$$\text{link}_{\tau} \Delta(g \cdot C) \cong \text{link}_{g \cdot \tau} (\Gamma(C)|_{([n] \setminus \sigma) \sqcup \bar{\sigma}}).$$

Hence, the conditions of the proposition are equivalent to the conditions of Definition 6. \square

3.2 Spherical Link Obstructions

Here we introduce another obstruction that can be detected via the polar complex of stable hyperplane codes. We use the following notation to aid our discussion. For a face $F \in \Gamma(C)$, we write $F = F^+ \sqcup \overline{F^-}$ to denote the restrictions of F to $[n]$ and $\overline{[n]}$. The *support* of F is $\underline{F} = F^+ \cup F^-$, the set of (barred or unbarred) vertices appearing in it.

For stable arrangements (\mathcal{H}, X) , Lemmas 1 and 2 allow us to translate between faces of $\Gamma(\text{code}(\mathcal{H}, X))$ and convex subsets of X as follows: The face $F = F^+ \sqcup \overline{F^-} \in \Gamma(C)$ corresponds to the open convex set

$$R_F = X \cap \left(\bigcap_{i \in F^+} H_i^+ \right) \cap \left(\bigcap_{j \in \overline{F^-}} H_j^- \right).$$

Note that for a facet $F = \sigma \sqcup \overline{([n] \setminus \sigma)}$ of the polar complex, R_F is precisely the interior of the atom A_σ . In addition, it is easy to see that $\text{link}_F \Gamma(C) = \Gamma(C')$ for some $C' \subseteq 2^{[n] \setminus E}$. Therefore, we consider the topology of the covered subset of R_F . We show the positive and negative halfspaces indexed by the complement of \underline{E} will cover either all of R_F or all but a linear subspace of R_F . The following proposition describes the combinatorics of the nerve of this cover.

Proposition 3 *Let (\mathcal{H}, X) be a stable arrangement, and let R_F be a nonempty region with $|F| < n$. Then $(\{H_i^+ \cap R_F\}_{i \notin \underline{E}}, R_F)$ is a stable arrangement. Moreover, the nerve $(\{H_i^+, H_i^-\}_{i \notin \underline{E}})$ is either collapsible or is the polar complex of the full code on the vertices $[n] \setminus \underline{E}$, i.e. $\text{nerve}(\{H_i^+, H_i^-\}_{i \notin \underline{E}}) = \Gamma(2^{[n] \setminus \underline{E}})$.*

Proof Denote $\nu \stackrel{\text{def}}{=} [n] \setminus \underline{E}$. First we verify the arrangement $(\{H_i^+ \cap R_F\}_{i \in \nu}, R_F)$ is stable. The region R_F is open and convex, and intersections of hyperplanes in R_F lie in X , so they already satisfied the genericity condition.

Consider $H_\nu \cap R_F$; if it is empty, then the union of the positive and negative open half-spaces indexed by ν is all of the convex set R_F , and so by Lemma 3 the nerve is collapsible. If $H_\nu \cap R_F \neq \emptyset$, by stability, we have $\dim H_\nu = d - |\nu|$. In this case, the linear independence of $\{w_i \mid i \in \nu\}$ ensures all of the $2^{|\nu|}$ intersection patterns of halfspaces, i.e. the nerve is $\Gamma(2^\nu) = \Gamma(2^{[n] \setminus \underline{E}})$. □

Definition 7 Let $F \in \Gamma(C)$ be a non-maximal face such that $\text{link}_F(\Gamma(C))$ is neither collapsible nor $\text{link}_F(\Gamma(C)) = \Gamma(2^{[n] \setminus \underline{E}})$. We call F a *sphere link obstruction*.

By Lemma 2, we have $\text{link}_F \Gamma(C) = \text{nerve}(\{H_i^+ \cap R_F, H_i^- \cap R_F\}_{i \notin \underline{E}})$. This, together with Proposition 3, imply

Theorem 2 (Sphere Link Property) *Suppose C is a stable hyperplane code. Then C has no sphere link obstructions.*

Example 4 Continuing Example 2, we consider the polar complex $\Gamma(C_2)$ for the non-stable arrangement (\mathcal{H}, X) in Fig. 2a. This complex is illustrated in Fig. 2c. The face \emptyset is a sphere link obstruction: $\text{link}_\emptyset \Gamma(C_2) = \Gamma(C_2)$, and this complex is neither the complex $\Gamma(2^{[3]})$, which would have 8 facets, nor is it collapsible. Therefore, C_2 is not a stable hyperplane code.

3.3 Chamber Obstructions

The intuition behind the third obstruction in this section concerns maximal hyperplane intersections. If a collection $\{H_i\}_{i \in \sigma}$ of hyperplanes intersects in a point ($\dim H_\sigma = 0$), then that point has fixed position relative to other hyperplanes. In particular, there cannot be two distinct regions defined by the other hyperplanes that contain that point. More generally, if $H_\sigma \neq \emptyset$ is a maximal non-empty intersection, then it intersects only one atom of the arrangement $\{H_j\}_{j \notin \sigma}$ of the remaining hyperplanes.

Definition 8 The *geometric chamber complex* of a hyperplane arrangement \mathcal{H} relative to an open convex set X , $\text{cham}(\mathcal{H}, X)$, is the set of $\sigma \subseteq [n]$ such that $H_\sigma \cap X \neq \emptyset$. By convention, $H_\emptyset = \mathbb{R}^d$ so $\emptyset \in \text{cham}(\mathcal{H}, X)$ for all (\mathcal{H}, X) .

The *combinatorial chamber complex* of a code C , denoted $\text{cham}(C)$, is given by the set of $\sigma \subseteq [n]$ such that there exists $T \in \Gamma(C)$ with $\underline{T} = [n] \setminus \sigma$ and $\text{link}_T \Gamma(C) = \Gamma(2^\sigma)$. We call such a subset T a *chamber* of σ .

Both $\text{cham}(\mathcal{H}, X)$ and $\text{cham}(C)$ are simplicial complexes: the former because for any $i \in \sigma$, $H_{\sigma \setminus i} \supseteq H_\sigma$; the latter because if $\text{link}_T \Gamma(C) = \Gamma(2^\sigma)$ then $\text{link}_{T \cup i} = \Gamma(2^{\sigma \setminus i})$. For stable hyperplane codes, the facets of these simplicial complexes correspond to maximal hyperplane intersections.

Example 5 Returning to the stable code C_1 from Example 1, the maximal faces of $\text{cham}(C_1)$ are 2 and 13. This is because $\text{link}_{1\bar{3}}(\Gamma(C_1)) = \Gamma(2^{(2)})$ and $\text{link}_2(\Gamma(C_1)) = \Gamma(2^{(1,3)})$. By inspection, these are also maximal faces of the geometric chamber complex $\text{cham}(\mathcal{H}, X)$ for the arrangement in Fig. 1a.

Proposition 4 For a stable arrangement (\mathcal{H}, X) , the associated chamber complexes coincide, $\text{cham}(\mathcal{H}, X) = \text{cham}(\text{code}(\mathcal{H}, X))$. Moreover, for $C = \text{code}(\mathcal{H}, X)$, each facet σ of $\text{cham}(C)$ has a unique chamber $T \in \Gamma(C)$.

Proof Let (\mathcal{H}, X) be a stable pair and set $C = \text{code}(\mathcal{H}, X)$. Suppose $\sigma \in \text{cham}(\mathcal{H}, X)$, so $H_\sigma \cap X \neq \emptyset$. Then, for any atom A_τ of the arrangement $(\{H_i^+ \cap X\}_{i \notin \sigma}, X)$ such that $H_\sigma \cap A_\tau \neq \emptyset$, the set $T = \tau \sqcup ([n] \setminus \sigma) \setminus \tau$ is a chamber of σ , hence $\sigma \in \text{cham}(C)$. For the reverse containment, suppose $\sigma \in \text{cham}(C)$ has chamber T . Then

$$\Gamma(2^\sigma) = \text{link}_T \Gamma(C) = \Gamma(\text{code}(\{H_i^+ \cap R_T\}_{i \notin \sigma}, R_T)),$$

meaning the hyperplanes $\{H_i\}_{i \in \sigma}$ partition R_T into the maximal number of regions, i.e. it is a central arrangement. Thus $H_\sigma \cap R_T \neq \emptyset$ and therefore $H_\sigma \cap X \neq \emptyset$ and $\sigma \in \text{cham}(\mathcal{H}, X)$.

Now consider σ a facet of $\text{cham}(C)$. Because $C = \text{code}(\mathcal{H}, X)$, the intersection of hyperplanes $H_\sigma \cap X$ does not meet any other hyperplanes inside X . Therefore, it is interior to only one atom of the arrangement $(\{H_j^+\}_{j \notin \sigma}, X)$; the face in $\Gamma(C)$ corresponding to this atom is the unique chamber T . \square

We reformulate Proposition 4 into our third and final obstruction to hyperplane codes.

Definition 9 Let $\sigma \subseteq [n]$ be a maximal face of $\text{cham}(C)$ such that there exist two faces $T_1 \neq T_2 \in \Gamma(C)$ with $\text{link}_{T_1} \Gamma(C) = \text{link}_{T_2} \Gamma(C) = \Gamma(2^\sigma)$. Then we call σ a *chamber obstruction*.

Theorem 3 (Single Chamber Property) Suppose $C = \text{code}(\mathcal{H}, X)$ is a stable hyperplane code. Then C has no chamber obstructions.

Example 6 The code C_3 from Example 3 also has a chamber obstruction, in the form of $\sigma = \{1, 2\}$. There are two faces $\{\bar{3}, 4\}$ and $\{3, \bar{4}\}$ with link in $\Gamma(C_3)$ equal to

the full polar complex on $\{1, 2\}$. One can check that this is maximal in $\text{cham}(C_3)$, creating a chamber obstruction.

4 The Main Results

Our main results consist of showing that (1) the polar complex of a stable hyperplane code is shellable and (2) shellability of $\Gamma(C)$ implies C has none of the obstructions thus far considered, except possibly the strong bitflip obstruction. First, we define shellability.

Definition 10 Let Δ be a pure simplicial complex of dimension d and F_1, \dots, F_t an ordering of its facets. The ordering is a *shelling order* if, for $i > 1$, the complex

$$\Delta(\{F_i\}) \cap \Delta(\{F_1, \dots, F_{i-1}\})$$

is pure of dimension $d - 1$. A simplicial complex is *shellable* if its facets permit a shelling order.

A shelling order constructs a simplicial complex one facet at a time in such a way that each new facet is glued along maximal faces of its boundary. The facets of $\Gamma(C)$ correspond to codewords of C , thus a shelling order of $\Gamma(C)$ corresponds to an ordering of the codewords. We explicitly construct such an order in Sect. 7.1 to prove Theorem 4.

Theorem 4 *Let $C \subseteq 2^{[n]}$ be a stable hyperplane code. Then $\Gamma(C)$ is shellable.*

It turns out that the structure of shellable polar complexes does not allow for many of the obstructions thus far considered.

Theorem 5 *Let $C \subseteq 2^{[n]}$ be a combinatorial code such that $\Gamma(C)$ is shellable. Then,*

1. C has no weak bitflip local obstructions,
2. C has no sphere link obstructions, and
3. C has no chamber obstructions.

Theorem 5 is proven in Sect. 7.2. Note the conclusion of Theorem 5.1 refers to weak local obstructions, highlighting the gap between the notions of collapsibility and contractibility.

5 Discussion

Hyperplane codes are a special class of convex codes that naturally arise as the output of a one-layer feedforward network [13]. Hyperplane codes are a proper¹ subclass of the open convex codes. We set out to find obstructions to being a hyperplane code, while focusing on stable hyperplane codes. There are two reasons for primarily considering the stable hyperplane codes: (1) they are ‘generic’ in that they are stable to small perturbations, and (2) they allow the action of the group of bitflips $(\mathbb{Z}_2)^n$. The second property makes it natural to consider the polar complex $\Gamma(C)$ of a code, because the combinatorics of the polar complex captures all the bitflip-invariant properties of the underlying stable hyperplane code. We have established the following relationships among the properties of the polar complex of the code. The necessary conditions for C being a stable hyperplane code,

$$\Gamma(C) \text{ is shellable} \iff C \text{ is a stable hyperplane code} \implies \begin{cases} C \text{ has no } \textit{strong} \text{ bitflip obstructions,} \\ C \text{ has no sphere link obstructions,} \\ C \text{ has no chamber obstructions.} \end{cases}$$

We have also established that almost all currently known necessary conditions follow from the shellability of the polar complex:

$$\Gamma(C) \text{ is shellable} \implies \begin{cases} C \text{ has no } \textit{weak} \text{ bitflip obstructions,} \\ C \text{ has no sphere link obstructions,} \\ C \text{ has no chamber obstructions.} \end{cases}$$

Note that the shellability of the polar complex implies the lack of *weak* bitflip obstructions, while a stable hyperplane code lacks *strong* bitflip obstructions. It is currently an open problem if the gap between the strong and the weak versions of the local obstructions is indeed a property of shellable polar complexes. Alternatively, codes with shellable polar complexes may also lack the strong bitflip obstructions. An example of a code whose polar complex is shellable, but has the strong bitflip obstruction² would provide a negative answer to the following open question: Is shellability of the polar complex equivalent to the code being a stable hyperplane code?

What makes a code a stable hyperplane code is still an open question. It seems likely that the shellability of the polar complex is not *the only* necessary condition for a code to be a stable hyperplane code. From a computational perspective, deciding if a given pure simplicial complex is shellable is known to be an NP-hard problem [14]. This likely means that answering the question of whether a given code

¹See e.g. Example 3 and Fig. 3.

²In particular, the appropriate link in Definition 6 is contractible, but not collapsible.

is produced by a one-layer network may be not computationally feasible. Ruling out that a given code is a hyperplane code may be less computationally intensive however, as it can rely on computing the Betti numbers of the free resolution of the Stanley-Reisner ideal of the polar complex, as illustrated in the following section.

6 Algebraic Signatures of a Hyperplane Code

Given a code C , how can we rule out that C is a stable hyperplane code? In this section, we show how the tools from computational commutative algebra can be used to detect sphere link obstructions via Stanley-Reisner theory.

6.1 The Neural and the Stanley-Reisner Ideal

The connections between neural codes and Stanley-Reisner theory were first developed in [9], and later expanded upon in [10, 12], and [16]. The key observation is that a code $C \subseteq 2^{[n]}$ can be considered as a set of points in $(\mathbb{F}_2)^n$, and the vanishing ideal I_C of that variety is a “pseudo-monomial ideal” with many similarities to a monomial ideal. In this section, we show that this connection can be made more explicit via the polar complex.

First, we state necessary prerequisites about the neural ring. Let \mathbb{F}_2 denote the field with two elements, and consider the polynomial ring $R \stackrel{\text{def}}{=} \mathbb{F}_2[x_1, \dots, x_n]$. A polynomial $f \in R$ can be considered as a function $f : 2^{[n]} \rightarrow \mathbb{F}_2$ by defining $f(\sigma)$ as the evaluation of f with $x_i = 1$ for $i \in \sigma$ and $x_i = 0$ for $i \notin \sigma$. Polynomials of the form

$$x^\sigma (1-x)^\tau \stackrel{\text{def}}{=} \prod_{i \in \sigma} x_i \prod_{j \in \tau} (1-x_j),$$

where $\sigma, \tau \subseteq [n]$, are said to be *pseudo-monomials*. Note that the pseudo-monomial $x^\sigma (1-x)^{[n] \setminus \sigma}$ evaluates to 1 if and only if the support of x equals σ ; such a pseudo-monomial is called the *indicator function* of σ .

Definition 11 ([9]) The *vanishing ideal* of a code $C \subseteq 2^{[n]}$ is the ideal of polynomials that vanish on all codewords of C ,

$$I_C \stackrel{\text{def}}{=} \{f \in R \mid f(\sigma) = 0 \text{ for all } \sigma \in C\}.$$

The *neural ideal* of C is the ideal generated by indicator functions of non-codewords,

$$J_C \stackrel{\text{def}}{=} \langle x^\sigma (1-x)^{[n] \setminus \sigma} \mid \sigma \notin C \rangle.$$

The *boolean ideal* of C is the ideal generated by the *boolean relations*, pseudo-monomials with $\sigma = \tau = i$,

$$\mathcal{B} \stackrel{\text{def}}{=} \langle x_i(1 - x_i) \mid i \in [n] \rangle.$$

Lemma 4 ([9, Lemma 3.2]) *Let C be a neural code. Then $I_C = J_C + \mathcal{B}$.*

Pseudomonomials in the vanishing ideal I_C correspond to relations of the form $\bigcap_{i \in \sigma} U_i \subseteq \bigcup_{j \in \tau} U_j$ among sets in any cover realizing C .

Lemma 5 ([9, Lemma 4.2]) *Let $C = \text{code}(\mathcal{U}, X)$ be a combinatorial code. Then*

$$x^\sigma(1 - x)^\tau \in I_C \iff \bigcap_{i \in \sigma} U_i \subseteq \bigcup_{j \in \tau} U_j,$$

where by convention $\bigcap_{i \in \emptyset} U_i = X$ and $\bigcup_{j \in \emptyset} U_j = \emptyset$.

In particular, the generators of \mathcal{B} correspond to the tautological relations $U_i \subseteq U_i$. The neural ideal records the non-tautological relations.

Definition 12 ([9]) A pseudo-monomial $f \in J_C$ is said to be *minimal* if there is no other pseudo-monomial $g \in J_C$ that divides f . The *canonical form* of J_C , denoted $CF(J_C)$, is the set of all the minimal pseudo-monomials in J_C .

The elements of the canonical form correspond to the minimal nontrivial relations $\bigcap_{i \in \sigma} U_i \subseteq \bigcup_{j \in \tau} U_j$. We will see that the canonical form of J_C and the Boolean relations also corresponds with the generating set of the Stanley-Reisner ideal of $\Gamma(C)$. We make these relationships explicit in Lemma 6 and Corollary 2.

The Stanley-Reisner correspondence associates to any simplicial complex on n vertices an ideal generated by square-free monomials in a polynomial ring in n variables [26]. The construction of the polar complex is seen to be particularly natural when considering its associated Stanley-Reisner ideal. For the unbarred vertices, we set the corresponding variables via $i \mapsto x_i$; for the barred vertices, we associate $\bar{i} \mapsto y_i$. The Stanley-Reisner ideal of $\Gamma(C)$ is the ideal in $S \stackrel{\text{def}}{=} \mathbb{F}_2[x_1, \dots, x_n, y_1, \dots, y_n]$ generated by the squarefree monomials indexed by *non-faces* of $\Gamma(C)$.

Definition 13 Let $C \subseteq 2^{[n]}$ be a combinatorial code. The Stanley-Reisner ideal of the polar complex is given by

$$I_{\Gamma(C)} = \langle x^\sigma y^\tau \mid \sigma \sqcup \bar{\tau} \notin \Gamma(C) \rangle \subseteq S.$$

Example 7 Consider the code $C_1 = \{1, 12, 123, 2, 23\}$ from Example 1. The corresponding variety in \mathbb{F}_2^3 is $\{100, 110, 111, 010, 011\}$ with canonical form given by

$$CF(J_{C_1}) = \{(1 - x_1)(1 - x_2), x_3(1 - x_2)\}.$$

The polar complex of C_1 is given by

$$\Gamma(C_1) = \Delta(\{\bar{1}\bar{2}\bar{3}, \bar{1}2\bar{3}, \bar{1}23, \bar{1}\bar{2}3, \bar{1}23\}).$$

The minimal nonfaces of $\Gamma(C_1)$ are $\{1\bar{1}, 2\bar{2}, 3\bar{3}, \bar{1}\bar{2}, \bar{2}\bar{3}\}$. This gives the Stanley-Reisner ideal

$$I_{\Gamma(C_1)} = \langle x_1y_1, x_2y_2, x_3y_3, y_1y_2, x_3y_2 \rangle.$$

The first three monomials in this list correspond to the Boolean relations, while the last two can be compared to the canonical form.

The intuition intimated by Example 7 holds true in general.

Lemma 6 *For any nonempty combinatorial code $C \subseteq 2^{[n]}$, the Stanley-Reisner ideal of the polar complex is induced by the canonical form and the Boolean relations. That is,*

$$x^\sigma y^\tau \in I_{\Gamma(C)} \iff x^\sigma(1-x)^\tau \in I_C. \tag{3}$$

and so

$$I_{\Gamma(C)} = \langle x^\sigma y^\tau \mid x^\sigma(1-x)^\tau \in CF(J_C) \rangle + \langle x_i y_i \mid i \in [n] \rangle. \tag{4}$$

Proof Consider a square-free monomial $x^\sigma y^\tau \in S$. By definition, $x^\sigma y^\tau \in I_{\Gamma(C)}$ if and only if $\sigma \sqcup \tau$ is a nonface of $\Gamma(C)$. The set $\sigma \sqcup \tau$ is a nonface of $\Gamma(C)$ if and only if any codeword in C which contains σ is not disjoint from τ , that is, C satisfies the following property:

$$\text{for all } \alpha \in C, \quad \sigma \subseteq \alpha \implies \alpha \cap \tau \neq \emptyset. \tag{5}$$

If C satisfies (5), the pseudomonomial $x^\sigma(1-x)^\tau$ vanishes on all of C , as x^σ evaluates to 0 on any codeword not containing σ , and $(1-x)^\tau$ evaluates to 0 on any codeword not disjoint from τ , e.g. any codeword containing σ . Conversely, if $x^\sigma(1-x)^\tau$ vanishes on all of C , every codeword that contains σ must not be disjoint from τ , so C satisfies (5). Therefore, $x^\sigma(1-x)^\tau \in I_C$. Thus we have established (3) and (4) follows, as any pseudomonomial in I_C is divisible either by $x_i(1-x_i)$ for some i , or by an element of the canonical form $CF(J_C)$. \square

The following is an immediate corollary of Lemmas 5 and 6.

Corollary 2 *Let $C = \text{code}(\mathcal{U}, X) \subseteq 2^{[n]}$ and $I_{\Gamma(C)}$ the Stanley-Reisner ideal of the polar complex of C . Then*

$$x^\sigma y^\tau \in I_{\Gamma(C)} \iff \bigcap_{i \in \sigma} U_i \subseteq \bigcup_{j \in \tau} U_j.$$

6.2 Sphere Link Obstructions and Multigraded Free Resolutions

In Sect. 3.2, we showed that $\text{link}_\Sigma(\Gamma(C))$ is either empty, collapsible, or is isomorphic to a sphere of dimension $n - |\Sigma| - 1$ when C is a stable hyperplane code. One consequence of this fact is that if a stable hyperplane realization of C exists, then a lower bound on the dimension of the realizing space is

$$d \geq \max_{\Sigma \in \Gamma(C)} \left\{ (n - |\Sigma|) \mid \text{link}_\Sigma(\Gamma(C)) \sim S^{n-|\Sigma|-1} \right\}.$$

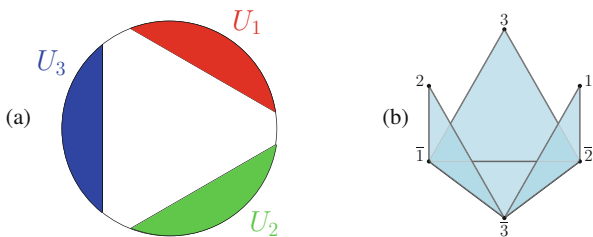
However, this may not be the true lower bound.

Example 8 Consider the code $C = \{\emptyset, 1, 2, 3\}$ consisting of four words; this can be realized by hyperplanes in \mathbb{R}^2 as in Fig. 4. Still, the polar complex $\Gamma(C)$ has facets $\overline{123}, \overline{12\bar{3}}, \overline{1\bar{2}3}, \overline{1\bar{2}\bar{3}}$, which has spherical links only at $\Sigma = \{\bar{i}, \bar{j}\}$ for $i \neq j \in \{1, 2, 3\}$. This might lead us to infer that the minimal realizing dimension is $n - |\Sigma| = 3 - 2 = 1$; however, it is easy to prove that it is impossible to realize by hyperplanes in \mathbb{R}^1 .

Another consequence of the sphere link property (Theorem 2) relates to algebraic properties of the Stanley-Reisner ring. The dual version of Hochster’s formula relates the multigraded minimal free resolution of the Stanley-Reisner ideal to the simplicial homology of the corresponding complex. A full exposition of minimal free resolutions is beyond the scope of this article, so we give a brief description and direct the reader to [23, Chapter 1] for more information.

The *multidegree* of a monomial $\left(\prod_{i=1}^n x_i^{a_i} \prod_{j=1}^n y_j^{b_j}\right) \in S$ is the vector of exponents $(a, b) = (a_1, \dots, a_n, b_1, \dots, b_n) \in \mathbb{N}^{2n}$. When the exponents are all 0 or 1, we identify the multidegree with its support as a subset of $[n] \sqcup [\bar{n}]$. The *coarse degree* of a monomial is the sum of the exponents $\sum_{i=1}^n a_i + \sum_{j=1}^n b_j \in \mathbb{N}$. For a homogeneous ideal $I \subset S$, a *minimal free resolution* of S/I is an exact sequence of free modules that terminates in $S/I \rightarrow 0$. Each module in the minimal free resolution of S/I can be *multigraded* so that each map in the resolution preserves multidegree. The *multigraded Betti number* of S/I , $\beta_{i,\sigma} = \beta_{i,\sigma}(S/I)$, is the rank of the free module in position i in the free resolution and with multidegree σ .

Fig. 4 (a) Realization of $C_4 = \{\emptyset, 1, 2, 3\}$ in \mathbb{R}^2 . Though sphere link dimension is 1, minimal realization dimension is 2. (b) The polar complex $\Gamma(C_4)$. The only non-collapsible links are of the form $\text{link}_{\bar{i}\bar{j}}\Gamma(C)$



Importantly for our purposes, these Betti numbers can be explicitly computed with `Macaulay2` [15] and similar computational algebra software.

Lemma 7 (Hochster’s Formula, Dual Version [23, Corollary 1.40]) For $\Gamma(C)$ the polar complex of a code $C \subseteq 2^{[n]}$ and Σ a face of $\Gamma(C)$,

$$\beta_{i+1, \Sigma^c}(S/I_{\Gamma(C)^\vee}) = \dim_k \tilde{H}_{i-1}(\text{link}_\Sigma \Gamma(C); k).$$

Here $\Sigma^c = ([n] \sqcup \overline{[n]}) \setminus \Sigma$ denotes the complement of Σ in the vertex set of $\Gamma(C)$, and $\Gamma(C)^\vee$ denotes the Alexander dual simplicial complex, $\Gamma(C)^\vee \stackrel{\text{def}}{=} \{F^c \mid F \notin \Gamma(C)\}$.

We use this lemma to detect sphere link obstructions.

Proposition 5 Let C be a stable hyperplane code with polar complex $\Gamma(C)$. Then, $\beta_{i, \sigma}(S/I_{\Gamma(C)^\vee}) = 0$ for all $i \geq 1$ except:

$$\begin{cases} \beta_{1, \Sigma^c}(S/I_{\Gamma(C)^\vee}) = 1 & \text{if } \Sigma \text{ is a facet.} \\ \beta_{n-|\Sigma|+1, \Sigma^c}(S/I_{\Gamma(C)^\vee}) = 1 & \text{if } \text{link}_\Sigma \Gamma(C) \sim S^{n-|\Sigma|-1}. \end{cases}$$

Proof Inserting $i = 0$ and Σ a facet into the dual version of Hochster’s formula yields

$$\beta_{1, \Sigma^c}(S/I_{\Gamma(C)^\vee}) = \dim_k \tilde{H}_{-1}(\text{link}_\Sigma \Gamma(C); k).$$

The right-hand side is equal to 1, since the link of a facet is the irrelevant simplicial complex, which gives a generator of (-1) -homology. This gives the first equation from the Proposition.

Setting $i = n - |\Sigma|$ and Σ a face of $\Gamma(C)$:

$$\beta_{n-|\Sigma|+1, \Sigma}(S/I_{\Gamma(C)^\vee}) = \dim_k \tilde{H}_{n-|\Sigma|-1}(\text{link}_{\Sigma^c} \Gamma(C); k).$$

The right-hand side is 1 precisely when the link is a sphere of the right dimension. In all other cases, the link is collapsible (Proposition 3) or equal to the void complex (links of non-faces), so the reduced homology is zero. \square

This proposition provides an algebraic signature of stable hyperplane codes.

Example 9 We again consider the code from Example 3. First, we translate into its polar complex $\Gamma(C_3)$, which has eleven facets for its eleven codewords. Then we compute the Stanley-Reisner ideal of its Alexander dual, and the Betti numbers associated to a minimal free resolution (e.g. using `Macaulay2`).

The table below is a condensed representation of the Betti numbers of $I_{\Gamma(C_3)}$, where the (i, j) -th entry is $\beta_{j, i+j}$ under the coarse grading.

$j \backslash i$	0	1	2	3	4
0	1				
1					
2					
3		11	16	6	
4			1	2	
5					1

The value of $\beta_{1,4}$ counts the codewords, which are facets of $\Gamma(C)$. The remaining entries of row 3 indicate links with the appropriate dimension. Rows 4 and 5, under the multigrading, point to the following nonzero Betti numbers:

$$\beta_{2,234\overline{134}} = 1, \quad \beta_{3,234\overline{1234}} = 1, \quad \beta_{3,1234\overline{134}} = 1, \quad \beta_{4,1234\overline{1234}} = 1.$$

Note that the multigrading of each Betti number corresponds to the link of its complement; specifically, $234\overline{134} \mapsto 1\bar{2}$, $234\overline{1234} \mapsto 1$, $1234\overline{134} \mapsto \bar{2}$, and $1234\overline{1234} \mapsto \emptyset$. These entries give us the following sphere link obstructions to $\Gamma(C_3)$ being the polar complex of a stable hyperplane code.

1. $\text{link}_{1\bar{2}}\Gamma(C_3) = \Delta(\{\bar{3}\bar{4}, \bar{3}4\})$, which has two connected components and hence nontrivial reduced homology of rank 1.
2. $\text{link}_1\Gamma(C_3) = \Delta(\{\bar{2}\bar{3}\bar{4}, \bar{2}3\bar{4}, \bar{2}\bar{3}4, \bar{2}34, \bar{2}\bar{3}4\}) \sim S^1$, which has the wrong dimension.
3. $\text{link}_{\bar{2}}\Gamma(C_3) = \Delta(\{\overline{134}, \bar{1}\bar{3}\bar{4}, \bar{1}3\bar{4}, 1\bar{3}\bar{4}\}) \sim S^1$, which also has the wrong dimension.
4. $\text{link}_{\emptyset}\Gamma(C_3) = \Gamma(C_3)$ has nontrivial homology, but $C_3 \neq 2^{[4]}$.

Each of these indicates the presence of a sphere link obstruction. Thus, C cannot be a stable hyperplane code.

7 Proofs of Theorems 4 and 5

Here we present the proofs of Theorems 4 and 5.

7.1 Shellability

The proof of Theorem 4 is organized as follows. First, we prove it in the special case $X = \mathbb{R}^d$. To extend the proof to the general case, we prove stable hyperplane codes can be realized by a pair $(\mathcal{H}, \mathcal{P})$ with \mathcal{P} the interior of a convex polyhedron with bounding hyperplanes \mathcal{B} such that $(\mathcal{H} \cup \mathcal{B}, \mathbb{R}^d)$ is a stable arrangement. Lastly, we use links to consider \mathcal{P} as a region in \mathbb{R}^d , reducing to the special case.

To prove the special case of Theorem 4, we use the following equivalent definition of a shelling order (see, for example, [26, Chapter III]).

Definition 14 Let Δ be a simplicial complex and F_1, \dots, F_t an ordering of its facets. The ordering is a shelling order if the sequence of complexes $\Delta_i = \Delta(\{F_1, \dots, F_i\})$, for each $i = 2, \dots, t$, satisfies the property that the collection of faces $\Delta_i \setminus \Delta_{i-1}$ has a *unique* minimal element, denoted $r(F_i)$ and called the *associated minimal face* of F_i .

Lemma 8 *If $(\mathcal{H}, \mathbb{R}^d)$ has generic intersections, then $\Gamma(\text{code}(\mathcal{H}, \mathbb{R}^d))$ is shellable.*

Proof Let $C = \text{code}(\mathcal{H}, \mathbb{R}^d)$ with $k = |C|$ the number of codewords. Without loss of generality, the w_i defining the hyperplanes H_i are unit vectors that span \mathbb{R}^d . Recall the notation

$$R_F = \bigcap_{i \in F} H_i^+ \cap \bigcap_{\bar{j} \in F} H_j^-$$

for $F \in \Gamma(C)$. Our proof proceeds by induction on d , the ambient dimension. An example of the $d = 2$ case is illustrated in Fig. 5.

The base case $d = 1$ is straightforward and guides the intuition for the general case. We order the codewords of C in a natural way based on their atoms, and show the corresponding ordering of facets of $\Gamma(C)$ is a shelling order. Each half-space H_i^+ is defined by an inequality of the form $x > h_i$ or $-x > h_i$ (i.e. $w_i = \pm 1$ for all i). Each atom A_σ has nonempty interior (a_σ, b_σ) with $a_\sigma = h_{i_\sigma}$ for some $i_\sigma \in [n]$, with one exception, where $a_\sigma = -\infty$. Order the codewords $\sigma_1, \dots, \sigma_k$ in increasing order of a_σ . This is a shelling order: when we add facet $\Sigma(\sigma)$ to our simplicial complex, this is the first time a facet contains i_σ if $w_i = 1$, otherwise it's the first time a facet contains \bar{i}_σ . In other words, σ is the first codeword in this order which contains i if $w_i = 1$ or the first codeword which does not contain i if $w_i = -1$; all later atoms lie on the same side of the hyperplane H_i . Thus, every facet of $\Gamma(C)$ has an associated minimal face and this ordering is a shelling order.

Now consider $d > 1$. Denote by $\Omega(\mathcal{H})$ the set of points where d hyperplanes intersect. We choose a generic “sweep” direction, a vector $u \in \mathbb{R}^d$ which satisfies the following properties:

- (i) u is not in the span of any $(d - 1)$ -element subset of $\{w_1, \dots, w_n\}$.
- (ii) For every pair of distinct points x, y in $\Omega(\mathcal{H})$, u is not in the orthogonal complement $(x - y)^\perp$.

Such a u exists because we exclude finitely many subsets of measure zero from \mathbb{R}^d . We use u to define a sliding hyperplane $H(t)$ and its corresponding “discovery time” function $m : C \rightarrow \mathbb{R} \cup \{-\infty\}$,

$$H(t) = \{x \in \mathbb{R}^d \mid u \cdot x - t = 0\}$$

$$m(\sigma) = \inf\{u \cdot x \mid x \in A_\sigma\}.$$

In the $d = 1$ case, $m(\sigma) = a_\sigma$ and thus induces a total order on codewords. For the $d > 1$ case, the goal is once again to use m to order the codewords. To do this, **(1)** we order the codewords with $m(\sigma) = -\infty$ inductively, then **(2)** we show m is injective on the remaining codewords, and lastly, **(3)** we show every facet has an associated minimal face.

(1) By construction, $\mathcal{H} \cup \{H^+(t)\}$ is a stable arrangement in \mathbb{R}^d for all but finitely many values of t , specifically, the values where $H(t)$ contains a point in $\Omega(\mathcal{H})$. Let t_0 be a constant less than all of these values (see Fig. 5a for an illustration). Property (i) ensures $H_i^+ \cap H(t_0) \neq \emptyset$ for all i , so in particular $\mathcal{L} \stackrel{\text{def}}{=} \{H_i^+ \cap H(t_0)\}$ is a stable arrangement in $H(t_0) \cong \mathbb{R}^{d-1}$. By inductive hypothesis, $\Gamma(\text{code}(\mathcal{L}, H(t_0)))$ is shellable. Each nonempty atom of the arrangement $(\mathcal{L}, H(t_0))$ is the intersection of an atom of $(\mathcal{H}, \mathbb{R}^d)$ with

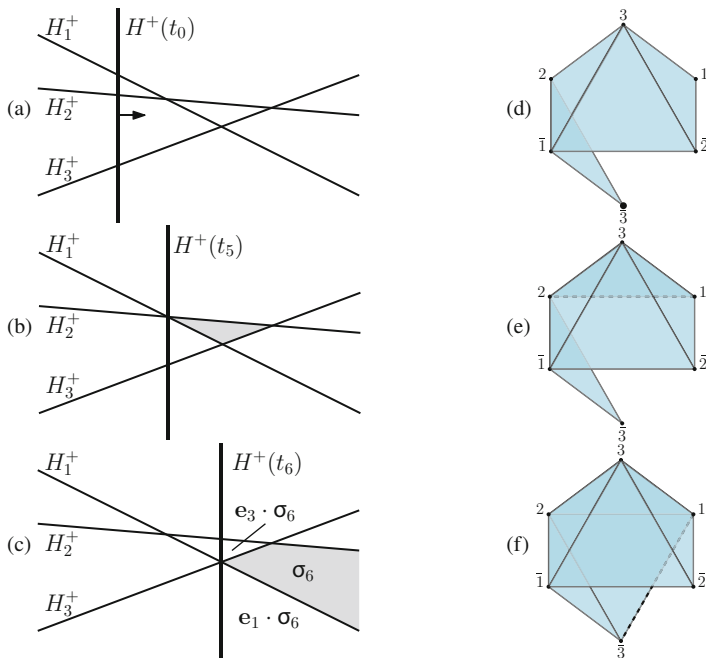


Fig. 5 An example of the shelling order construction in the $d = 2$ case. **(a)** The atoms discovered at time t_0 , i.e. the atoms A_σ with $m(\sigma) = -\infty$. Note the four atoms of $(\mathcal{H}, \mathbb{R}^2)$ which intersect $H(t_0)$ partition it into four intervals. **(b)** As t increases, $H(t)$ slides to the right, encountering atoms one at a time. The shaded atom is newly discovered. **(c)** Uniqueness of $r(\Sigma(\sigma_6))$ follows because $\mathbf{e}_3 \cdot \sigma_6$ and $\mathbf{e}_1 \cdot \sigma_6$ have already been discovered. **(d–f)** The inductive step and next two steps of the shelling order. The associated minimal face is highlighted with a large mark (panel **(d)**) or a dashed line (panels **(e, f)**). **(d)** The polar complex $\Gamma(\text{code}(\mathcal{L}, H(t_0)))$. Ordering the four codewords discovered in panel **(a)** from top to bottom yields $r(\overline{12\bar{3}}) = \bar{3}$. **(e)** Facet 123 is added when $H(t)$ contains the intersection $H_1 \cap H_2$ (panel **(b)**), thus $r(123) = 12$. **(f)** Atom A_{12} is discovered when $H(t)$ contains $H_1 \cap H_3$ (panel **(c)**). Thus $r(12\bar{3}) = \overline{13}$

$H(t_0)$, and the corresponding codewords are precisely those with $m(\sigma) = -\infty$. Thus, we have an ordering for these codewords which is an initial segment of a shelling of $\Gamma(C)$ (Fig. 5d).

- (2) Let $\sigma \in C$ be a codeword with $m(\sigma) > -\infty$. The function $f(x) = u \cdot x$ is minimized along a face of the (closure of) polyhedron $R_{\Sigma(\sigma)}$; property (i) ensures this face is a vertex, which is an element of $\Omega(\mathcal{H})$. Property (ii) ensures $f|_{\Omega(\mathcal{H})}$ is injective. Therefore, m induces a total order on codewords σ with $m(\sigma) > -\infty$. Let $\sigma_1, \dots, \sigma_k$ be the ordering of codewords of C obtained appending this ordering to the order from (1). We will show each facet has an associated minimal face to complete the proof.
- (3) Denote $\Gamma_i \stackrel{\text{def}}{=} \Gamma(\{\sigma_1, \dots, \sigma_i\})$ for $i = 1, \dots, k$. From (1), $r(\Sigma(\sigma_i))$ is defined whenever $m(\sigma_i) = -\infty$. So, let σ_i be a codeword with $t_i = m(\sigma_i) > -\infty$, meaning there is a vertex of $R_{\Sigma(\sigma_i)}$ minimizing f . This vertex is an element of $\Omega(\mathcal{H})$, i.e. it is the intersection H_{α_i} of d hyperplanes (see Fig. 5b, c). For $F \in \Gamma(C)$ and $\alpha \subseteq [n]$, we denote

$$F|\alpha \stackrel{\text{def}}{=} F \cap (\alpha \sqcup \bar{\alpha}),$$

the subset of F with support α . We claim $r(\Sigma(\sigma_i)) = \Sigma(\sigma_i)|\alpha_i$ (see Fig. 5e, f). The region $R_{\Sigma(\sigma_i)|\alpha_i}$ is a cone supported by $H(t_i)$, so this is the first codeword in our order with this exact combination of “on” and “off” vertices indexed by α_i . Thus, $\Sigma(\sigma_i)|\alpha_i \in \Gamma_i \setminus \Gamma_{i-1}$.

Now consider $F = \Sigma(\sigma_i)|\beta \in \Gamma_i \setminus \Gamma_{i-1}$. Suppose, for the sake of contradiction, $\beta \not\supseteq \alpha_i$, that is, there is some $\ell \in \alpha_i \setminus \beta$. Then $F \subseteq \Sigma(\mathbf{e}_\ell \cdot \sigma_i)$. Note $\mathbf{e}_\ell \cdot \sigma_i \in C$ since, by genericity, all 2^d possible regions around the point H_{α_i} produce codewords. However, since $H(t_i)$ intersects the interior of $R_{\Sigma(\mathbf{e}_\ell \cdot \sigma_i)}$, we have $m(\mathbf{e}_\ell \cdot \sigma_i) < m(\sigma_i)$ and therefore $\Sigma(\mathbf{e}_\ell \cdot \sigma_i) \in \Gamma_{i-1}$. We reach a contradiction, as this implies $F \in \Gamma_{i-1}$. Therefore, $r(\Sigma_i) = \Sigma_i|\alpha_i$ is the unique minimal face in $\Gamma_i \setminus \Gamma_{i-1}$. This completes the proof. \square

We now prove that a stable hyperplane code is a subset of codewords of a stable hyperplane arrangement in \mathbb{R}^d .

Lemma 9 *If C is a stable hyperplane code, then C can be realized by a stable pair $(\mathcal{H}, \mathcal{P})$ such that $\mathcal{P} = \bigcap_{j \in [m]} B_j^+$ is an open polytope with bounding hyperplanes \mathcal{B} such that $\mathcal{H} \cup \mathcal{B}$ has generic intersections in \mathbb{R}^d .*

Proof Let (\mathcal{H}, X) be a stable pair realizing C . By Lemma 1, we can perturb the hyperplanes \mathcal{H} to an arrangement \mathcal{H}' while preserving the atoms of the arrangement (\mathcal{H}, X) , i.e. $\text{code}(\mathcal{H}', X) = \text{code}(\mathcal{H}, X)$. Thus, C has a realization (\mathcal{H}', X) such that \mathcal{H}' has generic intersections outside of X as well.

Applying Lemma 1 again, we can choose a point p_σ in the interior of $A_\sigma^{\mathcal{H}'}$ for every $\sigma \in C$. Let \mathcal{P} be the interior of the convex hull of the set of points $\{p_\sigma \mid \sigma \in C\}$; by perturbing the points slightly we may assume \mathcal{P} is full-dimensional. Let $\mathcal{B} = \{B_{n+1}^+, \dots, B_{n+m}^+\}$ denote the bounding hyperplanes of this polytope, i.e.

$\mathcal{P} = \bigcap_{j=n+1}^{n+m} B_j^+$. Since $\mathcal{P} \subseteq X$, we conclude $\text{code}(\mathcal{H}', \mathcal{P}) \subseteq \text{code}(\mathcal{H}', X)$. Since we chose a points p_σ for every codeword of C , $\sigma \in C$ implies $A_\sigma^{\mathcal{H}'} \cap \mathcal{P} \neq \emptyset$ and therefore $\text{code}(\mathcal{H}', X) \subseteq \text{code}(\mathcal{H}', \mathcal{P})$. Thus we have $C = \text{code}(\mathcal{H}', \mathcal{P})$ and $(\mathcal{H}', \mathcal{P})$ is a stable arrangement.

The hyperplanes in $\mathcal{H}' \cup \mathcal{B}$ do not necessarily have generic intersections. Again, we apply Lemma 1: one can perturb each hyperplane in \mathcal{B} to hyperplanes \mathcal{B}' , so that these hyperplanes have generic intersections, yet the appropriate code is preserved, i.e. $C = \text{code}(\mathcal{H}', \mathcal{P}) = (\mathcal{H}', \mathcal{P}')$, where \mathcal{P}' is the open polyhedron $\mathcal{P}' = \bigcap_{B \in \mathcal{B}'} B^+$. This completes the proof. \square

We extend Lemma 8 to the general case with the following standard lemma [5].

Lemma 10 ([5, Proposition 10.14]) *Let Δ be a shellable simplicial complex. Then $\text{link}_\sigma \Delta$ is shellable for any $\sigma \in \Delta$, with shelling order induced from the shelling order of Δ .*

Proof of Theorem 4 By Lemma 9, C can be realized as $C = \text{code}(\mathcal{H}, \mathcal{P})$ with

$$\mathcal{P} = \bigcap_{j=n+1}^{n+m} B_j^+$$

an open polyhedron such that the arrangement $\mathcal{H} \cup \mathcal{B}$ has generic intersections in \mathbb{R}^d . Set $C' = \text{code}(\mathcal{H} \cup \mathcal{B}, \mathbb{R}^d)$, a code on vertex set $[n + m]$. By Lemma 8, $\Gamma(C')$ is shellable. Set $F = \{n + 1, \dots, n + m\} \in \Gamma(C')$. Then we have

$$\text{link}_F \Gamma(C') = \Gamma \left(\text{code} \left(\mathcal{H}, \bigcap_{j=n+1}^{n+m} B_j^+ \right) \right) = \Gamma(C).$$

By Lemma 10, as the link of a shellable complex, $\Gamma(C)$ is shellable. \square

7.2 Obstructions Following from Shellability

In general, shellable simplicial complexes are homotopy-equivalent to a wedge sum of spheres, where the number and dimension of the spheres correspond to the facets with $r(F) = F$ in some shelling order [22]. First we prove a stronger version of this statement for the polar complex of a code, which will be used throughout the proofs of all parts of Theorem 5. Note the condition of this lemma is intrinsic to the polar complex of the code and does not rely on any particular realization.

Lemma 11 *If $\Gamma(C)$ is shellable, then either $C = 2^{[n]}$ or $\Gamma(C)$ is collapsible.*

Proof We induct on the number of codewords of C . Let F_1, \dots, F_t be a shelling order of $\Gamma(C)$, with $\sigma_1, \dots, \sigma_t$ the corresponding order of codewords in C . For ease of notation, let $C' = \{\sigma_1, \dots, \sigma_{t-1}\}$ denote the first $t - 1$ codewords in this shelling order. By construction, $\Gamma(C')$ is shellable. Because it has one fewer codeword than C , it cannot be the full code and therefore, by inductive hypothesis, $\Gamma(C')$ is collapsible.

By definition, $r(F_t)$ is the unique minimal element of the collection $\Gamma(C) \setminus \Gamma(C')$ and hence the only facet that contains $r(F_t)$ is F_t . If $r(F_t) \subsetneq F_t$, then $(r(F_t), F_t)$ is a free pair, and $\Gamma(C) \searrow_{r(F_t)} \Gamma(C')$ which is collapsible.

In the case $r(F_t) = F_t$, we claim we must have $C = 2^{[n]}$. Suppose not, for the sake of contradiction, and let $\tau \in 2^{[n]} \setminus C$. Note $\Gamma(2^{[n]} \setminus \{\tau\})$ is homeomorphic to a closed $(n - 1)$ -ball (as it is a sphere missing top-dimensional open disc). Since $\Gamma(C')$ is a collapsible subcomplex of a simplicial complex, $\Gamma(C)$ is homotopy-equivalent to the quotient space $\Gamma(C) / \Gamma(C')$ (see [18, Proposition 0.17 and Proposition A.5]). Because $r(F_t) = F_t$, the boundary of the simplex $\Delta(\{F_t\})$ is contained in $\Gamma(C')$, and therefore $\Gamma(C) / \Gamma(C')$ is homotopy equivalent to S^{n-1} . We reach a contradiction, as $\Gamma(C) \subseteq \Gamma(2^{[n]} \setminus \{\tau\})$, but there is no embedding $S^{n-1} \hookrightarrow \mathbb{R}^{n-1}$ (see, e.g. [18, Corollary 2B.4]). Therefore, in this case we have $C = 2^{[n]}$. \square

To prove Theorem 5.1, we need one more lemma. Note that this lemma concerns with *contractibility* of certain subcomplexes, hence it can only be used to show C has no *weak* local obstructions.

Lemma 12 ([8, Lemma 4.4]) *Let Δ be a simplicial complex on vertex set V . Let $\alpha, \beta \in \Delta$ with $\alpha \cap \beta = \emptyset$, $\alpha \cup \beta \subsetneq V$, and $\text{link}_\alpha(\Delta|_{\alpha \cup \beta})$ not contractible. Then there exists $\alpha' \in \Delta$ such that (i) $\alpha' \supseteq \alpha$, (ii) $\alpha' \cap \beta = \emptyset$, and (iii) $\text{link}_{\alpha'}(\Delta)$ is not contractible.*

Proof of Theorem 5.1 Assume that the polar complex $\Gamma(C)$ is shellable. To show that C has no weak local obstructions, first suppose $\tau \in \Delta(C)$ and $\text{link}_\tau \Delta(C)$ is not contractible. We will show $\tau \in C$. Note that $\Delta(C) = \Gamma(C)|_{[n] \sqcup \emptyset}$, thus we apply Lemma 12 to the pair $\alpha = \tau \sqcup \emptyset$, $\beta = ([n] \setminus \tau) \sqcup \emptyset$ in the polar complex $\Gamma(C)$: there exists a face $T \in \Gamma(C)$ such that (i) $T = T^+ \sqcup \overline{T^-} \supseteq \tau \sqcup \emptyset$, (ii) $T \cap (([n] \setminus \tau) \sqcup \emptyset) = \emptyset$, and (iii) $\text{link}_T \Gamma(C)$ is not contractible. Statements (i) and (ii) together imply $T^+ = \tau$. Statement (iii) together with Lemma 11, implies $\text{link}_T \Gamma(C) = \Gamma(2^{[n] \setminus \underline{T}})$. Therefore this link contains the facet F consisting of all barred vertices in $[n] \setminus \underline{T}$. Thus $T \cup F = \tau \sqcup \overline{[n] \setminus \tau}$ is a face of $\Gamma(C)$ and therefore $\tau \in C$; hence τ cannot be a local obstruction.

For any $g \in (\mathbb{Z}_2)^n$, the above argument extends to $g \cdot C$ verbatim, since $\Gamma(g \cdot C) = g \cdot \Gamma(C)$, and $g \cdot \Gamma(C)$ is also shellable. Thus, C has no bitflip local obstructions. \square

Proof of Theorem 5.2 Links of $\Gamma(C)$ are polar complexes of a code on a smaller set of vertices, and links of shellable complexes are shellable (Lemma 10). Therefore, we can apply Lemma 11 to conclude $\text{link}_F \Gamma(C)$ is either collapsible or $\Gamma(2^{[n] \setminus \underline{E}})$ for any $F \in \Gamma(C)$. Thus, no face F can be a sphere link obstruction. \square

We use one final lemma to prove Theorem 5.3, which concerns faces of simplicial complexes with collapsible links.

Lemma 13 *Let Δ be a simplicial complex with $\alpha \in \Delta$ such that $\text{link}_\alpha \Delta$ is collapsible. Then $\Delta \searrow \text{del}_\alpha \Delta$.*

Proof Let $(\sigma_1, \tau_1), \dots, (\sigma_k, \tau_k)$ be the sequence of free pairs along which $\Delta_1 = \text{link}_\alpha \Delta$ is collapsed (in particular, $\sigma_k = \emptyset$), resulting in the sequence of simplicial complexes

$$\text{link}_\alpha \Delta = \Delta_1 \searrow_{\sigma_1} \Delta_2 \searrow_{\sigma_2} \cdots \searrow_{\sigma_k} \Delta_{k+1} = \{ \}.$$

Consider the sequence $(\sigma_1 \cup \alpha, \tau_1 \cup \alpha), \dots, (\sigma_k \cup \alpha, \tau_k \cup \alpha)$ in Δ . We claim, $(\sigma_1 \cup \alpha, \tau_1 \cup \alpha)$ is a free pair: $\sigma_1 \cup \alpha \subsetneq \tau_1 \cup \alpha$ and $\tau_1 \cup \alpha$ is a facet of Δ . If $\sigma_1 \cup \alpha \subseteq \tau'$ for some facet τ' , then $\tau' \setminus \alpha$ is a facet of $\text{link}_\alpha \Delta$ which contains σ_1 , hence $\tau' = \tau$. This argument can be repeated for the pair $(\sigma_2 \cup \alpha, \tau_2 \cup \alpha)$ in $\text{del}_{\sigma_1 \cup \alpha} \Delta$, and so on, to show that this is a sequence of free pairs in Δ . Thus, we have a sequence of collapses

$$\Delta \searrow_{\sigma_1 \cup \alpha} \cdots \searrow_{\sigma_k \cup \alpha} \text{del}_{\sigma_k \cup \alpha} \Delta.$$

Since $\sigma_k \cup \alpha = \alpha$, we have $\Delta \searrow \text{del}_\alpha \Delta$. □

Proof of Theorem 5.3 Assume the polar complex $\Gamma(C)$ is shellable. We demonstrate that if $\sigma \in \text{cham}(C)$ has more than one chamber, then σ is not maximal.

Suppose $T_1 \neq T_2$ are chambers of σ , that is

$$\text{link}_{T_1} \Gamma(C) = \text{link}_{T_2} \Gamma(C) = \Gamma(2^\sigma).$$

We will proceed by induction on $k = |T_1 \setminus T_2| > 0$. Since $\underline{T_1} = \underline{T_2} = [n] \setminus \sigma$, k is the number of indices where one T_i has a barred vertex and the other does not.

For the base case $k = 1$, suppose $T_1 \setminus T_2 = i$. Then

$$\text{link}_{T_1 \cap T_2} \Gamma(C) = \Gamma(2^{\sigma \cup \{i\}})$$

so $\sigma \cup i \in \text{cham}(C)$ and σ is not maximal.

Now suppose $|T_1 \setminus T_2| = k > 1$. We produce a face F such that $\text{link}_F \Gamma(C) = \Gamma(2^\sigma)$ and $|T_1 \setminus F| < k$, giving the induction step. Let $T = T_1 \cap T_2$, and consider $\text{link}_T \Gamma(C)$. This is a shellable subcomplex of $\Gamma(2^{[n] \setminus \underline{\sigma}})$; denote its corresponding code by C' . Let $T'_1 = T_1 \setminus T$ and $T'_2 = T_2 \setminus T$; by design these are disjoint with $|T'_1 \setminus T'_2| = |T'_1| = |T'_2| = k$ and $\text{link}_{T'_i} \Gamma(C') = \Gamma(2^\sigma)$ for $i = 1, 2$. Because they are disjoint, $\text{star}_{T'_1} \Gamma(C') \cup \text{star}_{T'_2} \Gamma(C')$ is a suspension of $\Gamma(2^\sigma)$, making it homotopy equivalent to $S^{|\sigma|}$.

Consider a face $F' \in \Gamma(C')$ such that $\underline{F'} = T'_1$. By construction, $\text{link}_{F'} \Gamma(C')$ is a subcomplex of $\Gamma(2^\sigma)$. If $\text{link}_{F'} \Gamma(C') \neq \Gamma(2^\sigma)$, then the link is collapsible by Lemmas 10 and 11; Lemma 13 implies that $\Gamma(C')$ collapses to $\text{del}_{F'} \Gamma(C')$.

There are $2^k - 2$ faces $F' \neq T_1, T_2$ with $\underline{F'} = \underline{T_1}$. If none of these F' had $\text{link}_{F'}\Gamma(C') = \Gamma(2^\sigma)$, this would lead to a contradiction: we would have a sequence of collapses

$$\Gamma(C') \searrow \text{star}_{T_1'}\Gamma(C') \cup \text{star}_{T_2'}\Gamma(C').$$

Since $\Gamma(C)$ is shellable, by Lemma 11 it is homotopy equivalent to S^{n-1} or is contractible. Collapsing preserves homotopy type, so we reach a contradiction.

Therefore, for one of these F' we must have $\text{link}_{F'}\Gamma(C') = \Gamma(2^\sigma)$. Thus $\text{link}_{F' \cup T}\Gamma(C) = \Gamma(2^\sigma)$ and so we have another face in $\Gamma(C)$ whose link yields $\Gamma(2^\sigma)$, namely $F = F' \cup T$. Since $|T_1 \setminus F| < k$, by induction σ is not maximal in $\text{cham}(C)$. Therefore, if σ is maximal in C , it must have a unique chamber, and thus C has no chamber obstructions. \square

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